1. A FINITE PROBABILITY SPACE CONSISTS OF TWO INGREDIENTS:

- A FINITE SAMPLE SPACE Ω (AND THE CORRESPONDING EVENTS IN $\mathcal{P}(\Omega)$)
- A PROBABILITY DISTRIBUTION FUNCTION $\mathbb{P}: \Omega \to \mathbb{R}$ ASSIGNING A PROBABILITY TO EACH SAMPLE $s \in \Omega$, SATISFYING: ① $\mathbb{P}(s) \geq 0$
  ② $\sum_{s \in \Omega} \mathbb{P}(s) = 1$.

[FOR AN EVENT $A \subset \Omega$, WE DEFINE $\mathbb{P}(A) = \sum_{s \in A} \mathbb{P}(s)$, SO ② SAYS $\mathbb{P}(\Omega) = 1$]

2. IF $(X, \mathbb{P}_X)$ AND $(Y, \mathbb{P}_Y)$ ARE FINITE PROBABILITY SPACES, WE CAN DEFINE A PROBABILITY DISTRIBUTION ON $X \times Y$ VIA $\boxed{\mathbb{P}(x,y) = \mathbb{P}_X(x) \cdot \mathbb{P}_Y(y)}$

THIS PROBABILITY SPACE'S SAMPLES (x,y) CONSIST OF A SAMPLE $x \in X$ FOLLOWED BY A SAMPLE $y \in Y$, WHICH CAN BE INTERPRETED AS A PAIR OF SAMPLES IN SUCCESSION.

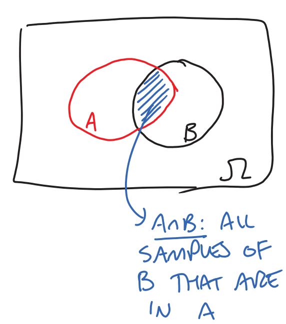
E.G., FLIP A FAIR COIN, (X) THEN ROLL A FAIR DIE: (Y)

X \ Y	1 $(\frac{1}{6})$	2 $(\frac{1}{6})$	3 $(\frac{1}{6})$	4 $(\frac{1}{6})$	5 $(\frac{1}{6})$	6 $(\frac{1}{6})$
$(\frac{1}{2})$ H	(H,1)	(H,2)	(H,3)	(H,4)	(H,5)	(H,6)
$(\frac{1}{2})$ T	(T,1)	(T,2)	(T,3)	(T,4)	(T,5)	(T,6)

ALL PROBABILITIES $\frac{1}{2} \cdot \frac{1}{6}$ HERE

3. IF $B \subset \Omega$ HAS $\mathbb{P}(B) > 0$:

(a) ANY EVENT $A \subset \Omega$ CAN BE INTERSECTED WITH B TO GIVE AN EVENT $A \cap B \subset B$; COLLECTING ALL OF THESE GIVES A SET OF EVENTS FOR THE SAMPLE SPACE B.



(b) FOR EACH EVENT $A \subset \Omega$, WE DEFINE $\boxed{\mathbb{P}_B(A) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}$

WE NEED TO DIVIDE BY $\mathbb{P}(B)$ IN ORDER TO ENSURE THAT $\mathbb{P}_B(B) = 1$, WHICH IS A REQUIREMENT FOR A PROBABILITY SPACE — WE CAN ONLY DO THIS IF $\mathbb{P}(B) > 0$!

4. IF A, B ARE EVENTS IN $(\Omega, \mathbb{P})$:

(a) A AND B ARE MUTUALLY EXCLUSIVE MEANS $\boxed{\mathbb{P}(A \cap B) = 0}$ (LITERALLY, THE PROBABILITY OF A AND B BOTH HAPPENING IS 0)

E.G.: ROLLING A FAIR DIE, A = ROLL A 3 OR 5
AND B = ROLL AN EVEN #
ARE MUTUALLY EXCLUSIVE, BECAUSE $A \cap B = \emptyset$, AND THUS $P(A \cap B) = 0$.

(b) A AND B ARE INDEPENDENT MEANS $\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)}$ (THE PROBABILITY OF BOTH HAPPENING IS EXACTLY THE PRODUCT OF EACH HAPPENING INDIVIDUALLY)

E.G.: ROLLING A FAIR DIE, A = ROLL A 1 OR 6
AND B = ROLL AN EVEN #
ARE INDEPENDENT: $\mathbb{P}(A) = 1/3$ (TWO SAMPLES, OF SIX TOTAL)
$\mathbb{P}(B) = 1/2$ (THREE SAMPLES, OF SIX TOTAL)
$\mathbb{P}(A \cap B) = 1/6 = \frac{1}{3} \cdot \frac{1}{2}$ (ONE SAMPLE OF SIX)

5. IF A, B ARE EVENTS IN $(\Omega, \mathbb{P})$ WITH $\mathbb{P}(B) > 0$:

(a) THE CONDITIONAL PROBABILITY $\boxed{P(A \underset{\text{"GIVEN"}}{|} B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}$

(b) THIS GIVES US THE PROBABILITY OF A HAPPENING, PRESUMING THAT B HAPPENS.

(c) THIS IS CORRECTLY CONCEPTUALIZED VIA SUBSPACES (SEE THE FORMULA IN 2(b)!):

- WITHIN THE SUBSPACE GIVEN BY B (I.E., SUPPOSING B HAPPENS)
- FIND $\mathbb{P}_B(A)$ (THE PROBABILITY THAT A HAPPEN, WITHIN THAT SUBSPACE)

(d) IF A AND B ARE INDEPENDENT, $\mathbb{P}(A \mid B) \overset{\text{DEF}}{=} \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \overset{*}{=} \dfrac{\mathbb{P}(A) \cdot \cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A)$.

I.E., THE CONDITIONAL PROBABILITY OF A GIVEN B IS EXACTLY THE SAME AS THE PROBABILITY OF A ALONE (A IS, LITERALLY, INDEPENDENT OF B!).

6. $(\Omega, \mathbb{P})$: ROLL TWO FAIR SIX-SIDED DICE.

(a) THE SET OF ALL PAIRS OF ROLLS IS SIMPLY THE CARTESIAN PRODUCT OF THE SET OF ROLLS FOR THE FIRST DIE AND THAT OF THE SECOND (WITH CORRESPONDING PROBABILITIES $\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$).

(b) (i) $\mathbb{P}(\text{FIRST ROLL EVEN}) = \frac{3\cdot 6}{36} = \frac{1}{2}$

(ii) $\mathbb{P}(\text{FIRST ROLL } 1, 2, \text{ OR } 3) = \frac{3\cdot 6}{36} = \frac{1}{2}$

(iii) $\mathbb{P}(\text{SECOND ROLL A MULTIPLE OF } 3) = \frac{6\cdot 2}{36} = \frac{1}{3}$

(iv) $\mathbb{P}(\text{SECOND ROLL } 1, 2, \text{ OR } 3) = \frac{6\cdot 3}{36} = \frac{1}{2}$

INDEPENDENT PAIRS:
- EITHER OF (i, ii) VS. EITHER OF (iii, iv) (COUNTING DIFFERENT DICE)
- (iii) VS. (iv): $\mathbb{P}(\text{SECOND ROLL: A MULTIPLE OF } 3 \wedge 1, 2, \text{ OR } 3)$
  $= \mathbb{P}(\text{SECOND ROLL} = 3) = \frac{6\cdot 1}{36} = \frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3}$
- NOT (i) VS. (ii): $\mathbb{P}(\text{FIRST ROLL: EVEN} \wedge 1, 2, \text{ OR } 3)$
  $= \mathbb{P}(\text{FIRST ROLL} = 2) = \frac{1\cdot 6}{36} = \frac{1}{6} \neq \frac{1}{2}\cdot\frac{1}{2}$

(c) B: SUM OF DICE 3, 4, 5, OR 6

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(i) Ω HAD 36 SAMPLES, EACH WITH PROBABILITY $\frac{1}{36}$.

OUR SUBSPACE B HAS 14 SAMPLES, SO $\mathbb{P}(B) = \frac{14}{36}$.

$\therefore$ PROBABILITIES OF THESE NEED TO BE SCALED BY $\frac{1}{\mathbb{P}(B)} = \frac{36}{14}$, GIVING EACH SAMPLE A PROBABILITY OF $\frac{1}{14}$.

(THIS IS ANOTHER UNIFORM DISTRIBUTION!)

(ii) (SINCE THE DISTRIBUTION IS UNIFORM, JUST COUNT SAMPLES IN THE K^{TH} ROW OR K^{TH} COLUMN:)

$$\mathbb{P}(A_1 | B) = \mathbb{P}_B(A_1) = \frac{4}{14} = \frac{2}{7}$$
$$\mathbb{P}(A_2 | B) = \mathbb{P}_B(A_2) = \frac{4}{14} = \frac{2}{7}$$
$$\mathbb{P}(A_3 | B) = \mathbb{P}_B(A_3) = \frac{3}{14}$$
$$\mathbb{P}(A_4 | B) = \mathbb{P}_B(A_4) = \frac{2}{14} = \frac{1}{7}$$
$$\mathbb{P}(A_5 | B) = \mathbb{P}_B(A_5) = \frac{1}{14}$$
$$\mathbb{P}(A_6 | B) = \mathbb{P}_B(A_6) = \frac{0}{14} = 0$$

A_6 & B ARE MUTUALLY EXCLUSIVE!

$\mathbb{P}(A_k) = \frac{1}{6}$ IN Ω, SO SINCE $\mathbb{P}(B) = \frac{14}{36} = \frac{7}{18}$, INDEPENDENCE WOULD REQUIRE A PROBABILITY OF $\frac{1}{6}\cdot\frac{7}{18} = \frac{7}{108}$
$\therefore$ NONE ARE INDEPENDENT OF B

(iii) IF A IS THE EVENT THAT BOTH ROLLS ARE EQUAL (SO $\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$),

$\mathbb{P}(A \cap B) = \frac{2}{36}$ (THE ONLY PAIRS ARE (2,2) & (3,3)).

$$\therefore \mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{2/36}{14/36} = \frac{2}{14} = \frac{1}{7}$$

(SUM 3,4,5,6 MEANS $\frac{1}{7}$ PROBABILITY THAT ROLLS ARE EQUAL)

$$\text{AND } \mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{2/36}{1/6} = \frac{12}{36} = \frac{1}{3}$$

(EQUAL ROLLS MEANS $\frac{1}{3}$ PROBABILITY THAT THE SUM IS 3, 4, 5, OR 6)

7. $(\Omega, \mathbb{P})$: TEN FLIPS OF A FAIR COIN; B: EXACTLY 5 FLIPS ARE H.

$\hookrightarrow \mathbb{P}(B) = \frac{\binom{10}{5}}{1024} = \frac{252}{1024}$ (CHOOSE 5 OF 10 TO BE H)

(a) A: FIRST FIVE FLIPS ARE H

$\hookrightarrow \mathbb{P}(A) = \frac{1^5 \cdot 2^5}{1024} = \frac{32}{1024}$ (NO CHOICE FOR FIRST 5; 2 CHOICES FOR THE REST)

$\mathbb{P}(A \cap B) = \frac{1}{1024}$ (IF THE FIRST 5 ARE H, THE REST MUST ALL BE T — ONLY ONE CHOICE!)

$\therefore \mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{1024} \cdot \frac{1024}{252} = \frac{1}{252}$ (5 HEADS MEANS $\frac{1}{252}$ CHANCE THAT THEY ARE THE FIRST 5)

AND $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{1}{1024} \cdot \frac{1024}{32} = \frac{1}{32}$. (FIRST FIVE ROLLS HEADS MEANS $1/32$ CHANCE THE REST ARE TAILS)

(b) (i) $\mathbb{P}_B(\text{FIRST FLIP } T) = \frac{1 \cdot \binom{9}{5}}{252}$ → (IF THE FIRST IS T, CHOOSE 5 H FROM THE OTHER 9)

{ NOTE THAT $\mathbb{P}(A \cap B)$ AND $\mathbb{P}(B)$ HAVE DENOMINATORS OF 1024, WHICH CANCEL; ALTERNATIVELY, B GIVES ANOTHER UNIFORM DISTRIBUTION, SO WE COULD JUST COUNT!)

$= \frac{126}{252} = \frac{1}{2}$

(ii) $\frac{1}{2}$ (THIS WORKS EXACTLY AS ABOVE, BUT FOR JUST THE SECOND FLIP)

(iii) $\mathbb{P}_B(\text{FIRST } \underline{\text{TWO}} \text{ FLIPS ARE } T) = \frac{1^2 \cdot \binom{8}{5}}{252}$ → AS IN (i): FIRST TWO FLIPS ARE DETERMINED, AND WE NEED 5 OF THE REMAINING 8 AS H.

$= \frac{56}{252} = \frac{14}{63}$ → A LITTLE LESS THAN $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

THESE TWO EVENTS — WHICH WERE "OBVIOUSLY" INDEPENDENT IN Ω — ARE NOT INDEPENDENT IN B!!

INTUITIVELY, THIS IS BECAUSE IF YOU KNOW THAT THE 10 ROLLS SPLIT 5-5 HEADS & TAILS, ANY GIVEN FLIP IS 50/50. BUT ONCE ONE FLIP IS T, WE NEED 5 H & 4 T AMONG THE REST, SO THE PROBABILITY OF A T TICKS DOWN A BIT.

NOTE THAT IN B, GIVEN THE EVENT A FROM PART (a), THERE WOULD BE 100% CHANCE OF THE LAST 5 FLIPS BEING T!

WHAT DO YOU THINK THE PROBABILITY OF THE THIRD FLIP BEING T WOULD BE IF THE FIRST TWO FLIPS WERE HT? (THE THIRD FLIP WOULD BE INDEPENDENT AGAIN!)

(c) (i) $\mathbb{P}_B(\text{FIRST THREE FLIPS } H) = \frac{1^3 \cdot \binom{7}{2}}{252}$ → SEE (b)(i, iii)

$= \frac{21}{252} = \frac{1}{12}$

(ii) $\frac{1}{12}$ (SAME ANALYSIS)

THESE TWO EVENTS ARE MUTUALLY EXCLUSIVE IN B, BECAUSE IN THE SUBSPACE B (5 HEADS!), IT'S IMPOSSIBLE TO HAVE THE FIRST 3 AND THE LAST THREE BE H.

(SPECIFICALLY: $\frac{1^3 \cdot \binom{4}{5} \cdot 1^3}{252} = 0$ BECAUSE $\binom{4}{5} = 0$)